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MATHEMATICAL MODELING OF THE QUASI-STATIONARY PROCESSES OF VISCOUS MIXTURE MIXING IN A RECTANGULAR AREA BY THE R-FUNCTIONS METHOD

Mixing processes are found in the chemical, pharmaceutical and food industries. Fluid mixing is one of the fundamental scientific problems associated with modern concepts of regular and chaotic dynamics. The paper considers the problem of mathematical modeling of the quasi-stationary process of mixing a viscous mixture. This problem consists of two sub-problems: determination of the velocity field in the flow region (Eulerian formalism) and investigation of the trajectories of individual fluid particles (Lagrange formalism). To solve the first subproblem, it is proposed to jointly use the principle of superposition, the structural method (method of R-functions) and the Ritz variational method. The methods of nonlinear dynamics and qualitative theory of differential equations are used to solve the second subproblem. A plane quasi-steady flow is considered in a rectangular region and it is assumed that the side walls are at rest, and the upper and lower walls move alternately according to the given laws. According to the method of R-functions, the structures of the solutions were built and the use of the Ritz variational method for the approximation of the uncertain components of the structures was justified. The operation of the proposed method is illustrated by the results of a computational experiment, which was conducted for different modes of wall motion. The practical interest of the considered regimes is due to the fact that they lead to the emergence of chaotic behavior when mixing occurs most efficiently. Using the methods of nonlinear dynamics, the location of periodic (hyperbolic and elliptical) points was investigated and the Poincaré section was constructed. Further research with the help of the method proposed in the work can be related to the consideration of flows in more geometrically complex regions and more complex mixing regimes, as well as in the application to the calculation of industrial problems.

Keywords: viscous fluid quasi-stationary flow, mixing flow, stream function, R-functions method, Ritz method, periodic points

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МАТЕМАТИЧНЕ МОДЕЛЮВАННЯ КВАЗІСТАЦІОНАРНИХ ПРОЦЕСІВ ПЕРЕМІШУВАННЯ В'ЯЗКОЇ СУМІШІ У ПРЯМОКУТНІЙ ОБЛАСТІ МЕТОДОМ *R*-ФУНКЦІЙ

Процеси перемішування зустрічаються в хімічній, фармацевтичній та харчовій промисловостях. Перемішування рідини є однією з фундаментальних наукових проблем, пов'язаних з сучасними концепціями регулярної та хаотичної динаміки. У роботі розглянуто задачу математичного моделювання квазістаціонарного процесу перемішування в'язкої суміші. Ця задача складається з двох підзадач: визначення поля швидкостей в області течії (формалізм Ейлера) та дослідження траєкторій окремих частинок рідини (формалізм Лагранжа). Для розв'язання першої підзадачі пропонується сумісно використати принцип суперпозиції, структурний метод (метод *R*-функцій) та варіаційний метод Рітца. Для розв'язання другої підзадачі використовуються методи нелінійної динаміки та якісної теорії диференціальних рівнянь. Плоска квазістаціонарна течія розглядається в прямокутній області і вважається, що бічні стінки перебувають у стані спокою, а верхня та нижня стінки рухаються по черзі за заданими законами. Відповідно до методу *R*-функцій побудовано структури розв'язків та обґрунтовано застосування для апроксимації невизначених компонент структур варіаційний метод Рітца. Роботу запропонованого методу проілюстровано результатами обчислювального експерименту, який було проведено для різних режимів руху стінок. Прикладний інтерес розглянутих режимів обумовлений тим, що вони призводять до виникнення хаотичної поведінки, коли перемішування відбувається найбільш ефективно. Методами нелінійної динаміки досліджено розташування періодичних (гіперболічних та еліптичних) точок та побудовано переріз Пуанкаре. Подальші дослідження за допомогою запропонованого у роботі методу можуть бути пов'язані з розглядом течій у більш геометрично складних областях та більш складних режимів перемішування, а також у застосуванні до розрахунку промислових задач.

Ключові слова: квазістаціонарна течія в'язкої рідини, перемішування, функція течії, метод *R*-функцій, метод Рітца, періодичні точки

Introduction. Mathematical modeling of viscous flows is widely used in investigation of mixing processes in chemical, pharmaceutical and food industries, etc. [1–3]. On the other hand, the problem of liquids mixing research is a fundamental scientific problem associated with chaotic dynamics [2–5]. There are several methods that are used for numerical simulation of these processes. But they do not have the universality property and can be only used to investigate processes in geometrically simple areas. In particular, in J. M. Ottino, H. Aref, V.V. Meleshko, T.A. Dunaeva, T.S. Krasnopolskaya [1, 6–8] and others works, the mixing problem was solved in a circle, a semicircle and a circular sector, etc. The methods proposed in these works cannot be applied to studying of mixing processes in more complex areas.

The geometric information included in the problem formulation can be taken into account accurately. This can be achieved by the usage of the constructive apparatus of the R-functions theory. The theory was proposed by V.L. Rvachev, Academician of the National Academy of Sciences of Ukraine [9]. Thus, it is a relevant scientific problem to develop new methods of numerical analysis of mixing processes based on the application of the R-functions theory. For instance, the R-functions method was applied to study the viscous fluid flows in [10–14]. This work continues the research initiated in [15, 16].

Problem statement. Let's consider a flat quasi-stationary flow of viscous incompressible fluid that fills Ω, the inside of a rectangle $\bar{\Omega} = [0, a] \times [0, b]$. We assume that the side walls $\bar{\Omega}$ are at rest and the top and bottom walls take turns move with speeds $\boldsymbol{v}_{\text{top}}(t) = (v_{\text{top}}(t), 0)$ and $\boldsymbol{v}_{\text{bot}}(t) = (v_{\text{bot}}(t), 0)$, respectively. Fig. 1 illustrates the scheme of such flow.

To solve the first part of the mixing problem we need to obtain the velocity field (v_x, v_y) in the flow area Ω. Let's assume that the considered flow is creeping. Then the nonlinear terms in the Navier – Stokes system of equations can be neglected and the flow can be described using the Stokes approximation [17].

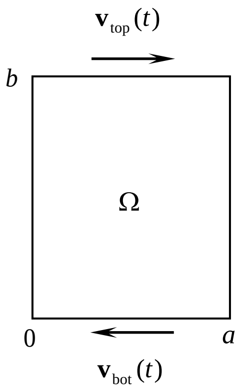

Fig. 1. Flow scheme

It is convenient to describe plane flows using the stream function $\psi(x, y, t)$ associated with the components of the velocity vector (v_x, v_y) by the relations [17]

$$v_x = \frac{\partial \psi}{\partial y}, \quad v_y = -\frac{\partial \psi}{\partial x}.$$

Under such conditions, we can set the following boundary-value problem for the stream function $\psi(x, y, t)$ in the Stokes approximation:

$$\Delta^2 \psi = 0 \ \text{ in } \ \Omega, \tag{1}$$

$$\psi\big|_{\partial\Omega} = 0,$$

$$\left.\frac{\partial \psi}{\partial \boldsymbol{n}}\right|_{\partial\Omega} = \begin{cases} -v_{\text{top}}(t), & (x, y) \in \partial\Omega_1 = \{y = b\}, \\ v_{\text{bot}}(t), & (x, y) \in \partial\Omega_3 = \{y = 0\}, \\ 0, & (x, y) \in \partial\Omega_2 = \{x = 0\} \cup \\ & \cup \partial\Omega_4 = \{x = a\}, \end{cases} \tag{2}$$

where $\boldsymbol{n}$ – is the external normal to the boundary $\partial\Omega$ of the flow area Ω;

Δ^2 – biharmonic operator,

$$\Delta^2 = \frac{\partial^4}{\partial x^4} + 2\frac{\partial^4}{\partial x^2 \partial y^2} + \frac{\partial^4}{\partial y^4}.$$

The solution of the second part consists of solving the initial problem for the equations system of Lagrangian particle motion:

$$\dot{x}(t) = \frac{\partial \psi(x, y, t)}{\partial y}, \ \dot{y}(t) = -\frac{\partial \psi(x, y, t)}{\partial x}, \tag{3}$$

$$x(t_0) = x_0, \ y(t_0) = y_0, \tag{4}$$

and constructing and analyzing the motion trajectories.

Basic Information on R-Function Theory. Let's consider the basics of the R-functions theory and the general application scheme of this theory methods in mathematical modeling of physical and mechanical fields [9].

A function $f : \mathbb{R}^n \to \mathbb{R}$ is called the R-function that corresponds to the three gradations partition of the set $\mathfrak{X} = (-\infty, +\infty) = \mathbb{R}$

$$S_3^{-1}(0) = \mathfrak{X}(0) = (-\infty, 0), \ \ S_3^{-1}(1) = \mathfrak{X}(1) = \{0\},$$

$$S_3^{-1}(2) = \mathfrak{X}(2) = (0, +\infty),$$

if such a three-valued logic function $Y = F(X_1, ..., X_n)$ exists, that

$$S_3[f(x_1, ..., x_n)] = F[S_3(x_1), ..., S_3(x_n)], \tag{5}$$

where

$$S_3(t) = \begin{cases} 0, & t < 0, \\ 1, & t = 0, \\ 2, & t > 0. \end{cases}$$

The three-valued logic function F that satisfies condition (5) is called accompanying for R-function f.

Thus, the definition (5) points out a functions class among the continuous argument functions. They have properties of the logic algebra functions (a discrete argument functions). In the R-functions theory, it is proved that this class has a nonempty intersection with the set of elementary functions. One of the most used in practical applications is the following R-functions system

$$\overline{\sigma} \equiv -\sigma; \ \ \sigma_1 \wedge_0 \sigma_2 \equiv \sigma_1 + \sigma_2 - \sqrt{\sigma_1^2 + \sigma_2^2};$$

$$\sigma_1 \vee_0 \sigma_2 \equiv \sigma_1 + \sigma_2 + \sqrt{\sigma_1^2 + \sigma_2^2}. \tag{6}$$

Accompanying functions for the R-functions of system (6) are three-valued negation, disjunction, and conjunction, respectively.

The application of the R-functions is primarily associated with an analytical description of geometric objects (solving the inverse problem of analytic geometry). It is also used to form structures for solving boundary value problems of mathematical physics (structural method).

One can describe an application scheme for the R-functions theory apparatus to solve the inverse problem of analytic geometry in the plane. Let a geometric object $\Omega \subset \mathbb{R}^2$ be constructed from support sets $\Sigma_i = (\sigma_i(x, y) \geq 0)$, $i = 1, ..., m$, using algebra of logic operations $-$, $\wedge$, $\vee$ in this form

$$\Omega = F(\Sigma_1, ..., \Sigma_m). \tag{7}$$

We assume that $\sigma_i(x, y)$, $i = 1, ..., m$, are simple continuous (elementary) functions, i.e. at the same time $\sigma_i(x, y) = 0$ is the boundary of sets $\sigma_i(x, y) \geq 0$ and $\sigma_i(x, y) > 0$. If in (7) we carry out a formal substitution for set Ω with $\omega(x, y)$, Σ_i with $\sigma_i(x, y)$, and the logic algebra symbols $-$, $\wedge$, $\vee$ with the corresponding R-functions symbols, we obtain an elementary function $\omega(x, y)$ in the form of a single analytical expression that is equal to zero on the boundary $\partial\Omega$. Moreover, the inequality $\omega(x,y)>0$ is true for the internal points Ω.

Thus, the implicit equation $\omega(x,y)=0$ defines the locus, which is the boundary of Ω.

Also, this function $\omega(x,y)$ can be subordinated to the normalization conditions:

$$\omega(x,y)=0 \text{ at } \partial\Omega;\ \omega(x,y)>0 \text{ in } \Omega;$$

$$\left.\frac{\partial\omega}{\partial\boldsymbol{n}}\right|_{\partial\Omega}=-1, \tag{8}$$

where $\boldsymbol{n}$ – external normal to $\partial\Omega$.

For boundary-value and initial-boundary-value problems of mathematical physics, the R-function method allows to construct general structures for solving. These structures are bundles of functions that exactly satisfy all the boundary conditions of the problem.

The application of the R-functions structural method for the numerical study of physical-mechanical fields consists of the following steps:

1) an exact analytical geometric description of the area Ω in which the field is considered, i.e. the construction of a function $\omega(x,y)$ with properties (8);

2) continuation of the boundary conditions into the area, i.e. until the functions and operators become defined on the boundary at internal points of the area Ω;

3) the construction of the general solution structure, i.e. such formula that depends on one or more indefinite functions and which, with any choice, exactly satisfies all boundary problem conditions;

4) the construction of an approximate solution, i.e. the approximation of the indeterminate components of the structure by some numerical method (grid method, Ritz method, Galerkin method, etc.).

The following two approaches are the most common ways to extend the boundary conditions within the area [9].

Let the function φ_0 at the points $\partial\Omega$ be defined as composite like this

$$\varphi_0(s)=\begin{cases}\varphi_0^{(1)}(s), & s\in\partial\Omega_1,\\ \dots & \dots\\ \varphi_0^{(r)}(s), & s\in\partial\Omega_r,\end{cases}$$

where the border sections $\partial\Omega_1$, …, $\partial\Omega_r$ are pairwise different, they do not have common internal points and $\partial\Omega=\partial\Omega_1\cup\ldots\cup\partial\Omega_r$.

If functions $\varphi_i(x,y)$, $i=1,...,r$, are such that $\left.\varphi_i\right|_{\partial\Omega_i}=\varphi_0^{(i)}$, and $\omega_i(x,y)$, $i=1,...,r$, such that $\omega_i(x,y)=0$ at $\partial\Omega_i$ and $\omega_i(x,y)>0$ in $\Omega\cup(\partial\Omega\setminus\partial\Omega_i)$, then the function

$$\varphi=\frac{\varphi_1\omega_2\omega_3...\omega_r+\varphi_2\omega_1\omega_3...\omega_r+...+\varphi_r\omega_1\omega_3...\omega_r}{\omega_2\omega_3...\omega_r+\omega_1\omega_3...\omega_r+...+\omega_1\omega_3...\omega_r} \tag{9}$$

has a property $\left.\varphi\right|_{\partial\Omega}=\varphi_0$. We will mark this continuation as $\mathrm{EC}\,\varphi_0=\varphi$.

The second approach is connected to the continuation of the differential operators into the area Ω. These operators are defined on the boundary $\partial\Omega$. Let $\omega(x,y)=0$ be the normalized equation of the boundary $\partial\Omega$ of the area Ω (i.e. the function $\omega(x,y)$ satisfies conditions (8)). Then the operator D_1, that behaves according to the rule

$$D_1u\equiv\frac{\partial\omega}{\partial x}\frac{\partial u}{\partial x}+\frac{\partial\omega}{\partial y}\frac{\partial u}{\partial y}, \tag{10}$$

at regular points of $\partial\Omega$ satisfies the equality

$$-\left.D_1u\right|_{\partial\Omega}=\frac{\partial u}{\partial\boldsymbol{n}},$$

where $\boldsymbol{n}$ – external normal to $\partial\Omega$. The expression D_1u makes sense everywhere in $\Omega\cup\partial\Omega$.

In particular [10], the structure of the problem solution

$$\Delta^2u=F \text{ in } \Omega, \tag{11}$$

$$\left.u\right|_{\partial\Omega}=\tilde f(s),\ \left.\frac{\partial u}{\partial\boldsymbol{n}}\right|_{\partial\Omega}=\tilde g(s),\ s\in\partial\Omega, \tag{12}$$

can be represented as follows

$$u=f-\omega(D_1f+g)+\omega^2\Phi, \tag{13}$$

where $f=\mathrm{EC}\,\tilde f$ and $g=\mathrm{EC}\,\tilde g$ – are extensions of functions $\tilde f$ and $\tilde g$ in Ω which are constructed using formula (9);

the operator D_1 is determined by equality (10);

Φ is the indefinite component of the structure, and the function $\omega(x,y)$ satisfies conditions (8).

Method of Numerical Analysis of Quasi-Stationary Viscous Fluid. In accordance with the principle of superposition, the solution of problem (1), (2) is represented in the form

$$\psi(x,y,t)=-v_{\text{top}}(t)\psi_1(x,y)+v_{\text{bot}}(t)\psi_2(x,y), \tag{14}$$

where $\psi_1(x,y)$ – is the solution of the problem

$$\Delta^2\psi_1=0 \text{ in } \Omega, \tag{15}$$

$$\left.\psi_1\right|_{\partial\Omega}=0,$$

$$\left.\frac{\partial\psi_1}{\partial\boldsymbol{n}}\right|_{\partial\Omega}=\begin{cases}1, & (x,y)\in\partial\Omega_1,\\ 0, & (x,y)\in\partial\Omega_2\cup\partial\Omega_3\cup\partial\Omega_4,\end{cases} \tag{16}$$

and $\psi_2(x,y)$ – is the solution of the problem

$$\Delta^2\psi_2=0 \text{ in } \Omega, \tag{17}$$

$$\left.\psi_2\right|_{\partial\Omega}=0,$$

$$\left.\frac{\partial \psi_2}{\partial \boldsymbol{n}}\right|_{\partial\Omega} = \begin{cases} 1, & (x, y) \in \partial\Omega_3, \\ 0, & (x, y) \in \partial\Omega_1 \cup \partial\Omega_2 \cup \partial\Omega_4. \end{cases} \quad (18)$$

To solve problems (15), (16) and (17), (18), we use the R-functions method and Ritz method.

Let's introduce the following functions

$$\omega_1 = b - y, \quad \omega_2 = x, \quad \omega_3 = y, \quad \omega_4 = a - x.$$

Each of these functions $\omega_i(x, y)$, $i = 1, 2, 3, 4$, has properties:

1) $\omega_i = 0$ at $\partial\Omega_i$;

2) $\omega_i > 0$ in $\Omega \cup (\partial\Omega \setminus \partial\Omega_i)$;

3) $\frac{\partial \omega_i}{\partial \boldsymbol{n}} = -1$ at $\partial\Omega_i$, $\boldsymbol{n}$ – external normal to $\partial\Omega_i$.

Then the function

$$\omega = \left[\frac{1}{a} x(a - x)\right] \wedge_0 \left[\frac{1}{b} y(b - y)\right], \quad (19)$$

where $\wedge_0$ – the R-conjunction sign, satisfies the conditions (8).

Each of problems (15), (16) and (17), (18) can be reduced to the form (11), (12) if (according to (9)) we choose

$$f_1 = 0, \quad f_2 = 0,$$

$$g_1 = \frac{\omega_2 \wedge_0 \omega_3 \wedge_0 \omega_4}{\omega_1 + \omega_2 \wedge_0 \omega_3 \wedge_0 \omega_4}, \quad g_2 = \frac{\omega_1 \wedge_0 \omega_2 \wedge_0 \omega_4}{\omega_3 + \omega_1 \wedge_0 \omega_2 \wedge_0 \omega_4}.$$

Then by formula (13) we obtain the solution structures for the problems (15), (16) and (17), (18) in the form

$$\psi_i = -\omega g_i + \omega^2 \Phi_i, \; i = 1, 2, \quad (20)$$

where the function ω is (19);

Φ_1, Φ_2 are the indeterminate components of the structures.

To approximate the indeterminate components in (20) we use Ritz's variational method [18]. To do this, in problems (15), (16), and (17), (18) we first make substitutions

$$\psi_i = h_i + u_i, \quad (21)$$

where $h_i = -\omega g_i$, $u_i = \omega^2 \Phi_i$ – new functions we are looking for, $i = 1, 2$. Substituting (21) into (15), (16), and (17), (18) for functions u_i, $i = 1, 2$, we obtain problems with homogeneous boundary conditions:

$$\Delta^2 u_i = -\Delta^2 h_i \text{ in } \Omega, \quad (22)$$

$$u_i\big|_{\partial\Omega} = 0, \; \left.\frac{\partial u_i}{\partial \boldsymbol{n}}\right|_{\partial\Omega} = 0. \quad (23)$$

We consider boundary value problems (22), (23) in Hilbert space $L_2(\Omega)$. We will associate each of these problems with an operator $A = \Delta^2$ at the function definition area

$$D_A = \left\{ u \,\middle|\, u \in C^4(\Omega) \cap C^1(\bar{\Omega}), u\big|_{\partial\Omega} = \left.\frac{\partial u}{\partial \boldsymbol{n}}\right|_{\partial\Omega} = 0 \right\} \subset L_2(\Omega).$$

The operator A defined in this way is symmetric and positively defined. If we close the set D_A in the norm generated by the scalar product

$$[u, v] = \iint_{\Omega} \Delta u \Delta v dx dy,$$

then we get the energy space H_A. Then, by the theorem on the energy functional [18] (provided that $\Delta h_i \in L_2(\Omega)$, $i = 1, 2$) the generalized solution of problems (22), (23) is

$$u_i = \arg \inf_{u \in H_A} \iint_{\Omega} [(\Delta u)^2 + 2\Delta u \Delta h_i] dx dy, \; i = 1, 2.$$

According to the Ritz method [18], approximate solutions of these variational problems will be sought in the form

$$u_i = \omega^2 \Phi_i \approx u_{i,N} = \omega^2 \Phi_{i,N} = \omega^2 \sum_{k=1}^{N} c_k^{(i)} \tau_k = \sum_{k=1}^{N} c_k^{(i)} \varphi_k,$$

where $i = 1, 2$, $\{\tau_k\}$ – any complete in $L_2(\Omega)$ system of functions (degree or trigonometric polynomials, splines, etc.), $\varphi_k = \omega^2 \tau_k$.

The sequence $\{\varphi_k\}$ is coordinate because

1) $\varphi_j \in D_A$ for anyone j;

2) $\varphi_1, ..., \varphi_N$ linearly independent for each N;

3) the system $\{\varphi_j\}$ is full in space H_A.

To determine the unknown numbers $c_1^{(i)}$, …, $c_N^{(i)}$, $i = 1, 2$, it is necessary to solve the Ritz system

$$\sum_{k=1}^{N} [\varphi_k, \varphi_j] c_k^{(i)} = -(\Delta h_i, \Delta \varphi_j), \quad (24)$$

$$j = 1, ..., N, \; i = 1, 2,$$

where

$$[\varphi_k, \varphi_j] = \iint_{\Omega} \Delta \varphi_k \Delta \varphi_j dx dy, \; k, j = 1, ..., N,$$

$$(\Delta h_i, \Delta \varphi_j) = \iint_{\Omega} \Delta h_i \Delta \varphi_j dx dy, \; j = 1, ..., N, \; i = 1, 2.$$

The matrices of systems (24) are independent of i and can be calculated once, when we solve problems (22), (23). In addition, these matrices are symmetric, which also reduces the amount of computation.

It follows from the Ritz method convergence theorem [18] that the sequences of functions $u_{i,N}$, $i = 1, 2$, $N \to \infty$, converge to the unique generalized solutions of boundary value problems (22), (23) in both norms $L_2(\Omega)$ and H_A. Then the sequences of functions $\psi_{i,N} = h_i + u_{i,N}$, $i = 1, 2$, converge in norm $L_2(\Omega)$ to the unique generalized solutions of problems (15), (16) and (17), (18) and the sequence of functions

$$\psi_N(x,y,t) = -v_{\text{top}}(t)\psi_{1,N}(x,y) + v_{\text{bot}}(t)\psi_{2,N}(x,y)$$

converges to the unique generalized solution of the original problem (1), (2).

So the first part of the mixing problem is solved.

To solve the second part of the mixing problem we need to study the methods of qualitative theory of differential equations of the initial problem

$$\dot{x}(t) = \frac{\partial\psi_N(x,y,t)}{\partial y},\ \dot{y}(t) = -\frac{\partial\psi_N(x,y,t)}{\partial x}, \qquad (25)$$

$$x(t_0) = x_0,\ y(t_0) = y_0, \qquad (26)$$

that describes the behavior of individual fluid particles (markers).

In considered problem (25), (26), it is necessary to find the periodic points of system (25), find out their nature, study the behavior of phase trajectories, etc. [5].

Results. The computational experiment was performed for a rectangular area for different ratios $a:b$ and for different movement modes of the top and bottom walls. The integrals in system (24) were numerically calculated using a Gaussian cubic formula. The fifth-degree Schönberg splines for the system $\{\tau_k\}$ were chosen. Two quasi-stationary modes (T – period) were considered:

– mode A: alternately at time intervals $\left[kT; \frac{T}{2}+kT\right]$, $\left[\frac{T}{2}+kT; T+kT\right]$, $k \in \mathbb{N}$, the top and bottom walls move respectively at regular intervals in opposite directions at constant speeds;

– mode B: the top wall moves at the time interval $\left[kT-\varepsilon; \frac{T}{2}+\varepsilon+kT\right]$, $k \in \mathbb{N}$, and the bottom wall moves in the opposite direction at the time interval $\left[\frac{T}{2}-\varepsilon+kT; T+kT+\varepsilon\right]$, $k \in \mathbb{N}$ $(0 < \varepsilon < \frac{T}{2})$.

The quasi-stationary modes under consideration are of applied interest. In these cases, chaotic behavior may occur when mixing is the most effective. In the case of stationary mode, the phase trajectory of markers in the area Ω is a closed curve without self-intersection points. For quasi-stationary mode the phase trajectory has more complex behavior. Fig. 3 shows the marker trajectory exiting from the point $(0.52; 0.53)$ in 10 periods in case of mode B.

Periodic points for different ratios $a:b$ were numerically found for the considered modes. Schemes of their location in the area Ω are shown in Fig. 3 and 4 (✕ – hyperbolic point; • – elliptic point). Poincare Sections for 300 points were constructed as well. The obtained cross sections allow us to conclude that in quasi-stationary mode A the regularity of the flow is disturbed. In mode B there is an area of global chaos with the exception of only the border zones. For the area Ω when $a=1$, $b=1$, their examples for different modes are shown in Fig. 5 and 6. The obtained results are in good agreement with the results of physical experiments [1, 5] and with the results obtained by means of other methods [4, 8].

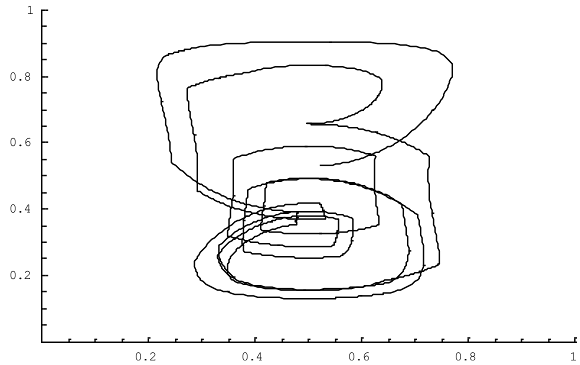

Fig. 2 The marker trajectory exiting from the point $(0.52; 0.53)$ in 10 periods (mode B)

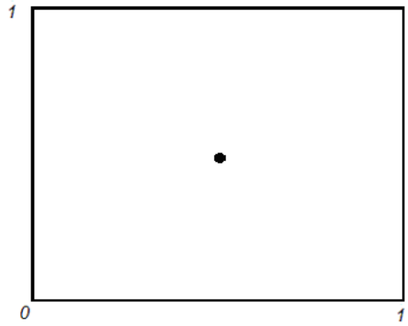

Fig. 3 Location of periodic points in the area Ω ($a=1$, $b=1$, modes A and B)

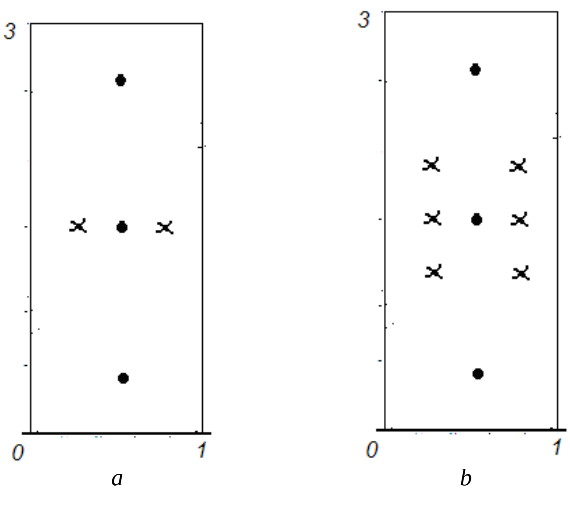

Fig. 4 Location of periodic points in area Ω ($a=1$, $b=3$) mode A (*a*) and mode B (*b*)

Conclusions. The technique for investigating the processes of mixing viscous mixtures was proposed. It is based on the joint application of the structural method (*R*-functions method) and methods of qualitative theory of differential equations. The application of the *R*-functions method made it possible to obtain the analytic solution of the first part of the mixing problem – viscous stream flow function. That facilitated its further usage for solving the second part of the mixing problem – to study the properties of the trajectories of the marker movements.

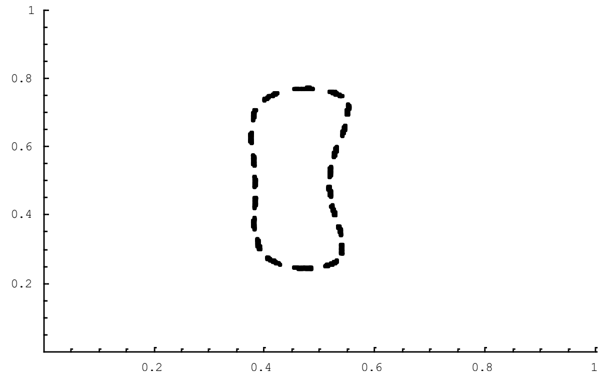

Fig. 5 Poincare Intersection of the trajectory exiting from the point $(0.52;\ 0.53)$ (mode A)

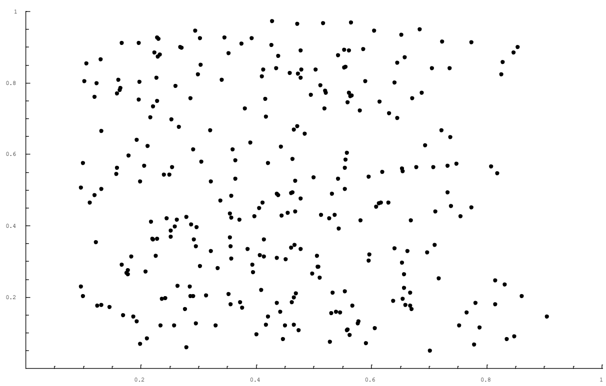

Fig. 6 Poincare Intersection of the trajectory exiting from the point $(0,52;\ 0,53)$ (mode B)

The application of the *R*-functions method allows to perform computational experiments for more complex areas than those found in modern research. That made the proposed method more universal than the known ones. Experimental studies of two of mixing modes allowed us to make conclusions about their effectiveness. Further studies of the proposed method may be related to the consideration of flows in more geometrically complex areas and more complex mixing modes.

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